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VARIATIONAL PRINCIPLES OF THE FILTRATION OF AN INCOMPRESSIBLE FLUID IN MEDIA WITH DUAL POROSITY†

P. A. MAZUROV

Kazan

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Dual variational principles for the steady single-phase and two-phase filtration of an incompressible fluid in media with dual porosity are constructed. By a medium with dual porosity, we mean two media, which are embedded in one another, with different porosities and permeabilities and which are coupled by a fluid crossflow [1]. The principle of minimum energy dissipation [2, 3], which is also used to close the equations and to find the structure of generalized forces, is the basis for constructing the variational principles. These principles enable one to determine the pressure fields, crossflows and filtration rates in the media. For steady single-phase filtration, the variational principles completely define the solution of the problem while, in the case of two-phase filtration, they hold for fixed saturations.

1. WE SHALL construct variational principles in the case of steady single-phase filtration and write the continuity equations in the media f and p as

$$\operatorname{div} \mathbf{q}_f = Q, \quad \operatorname{div} \mathbf{q}_p = -Q \tag{1.1}$$

where $\mathbf{q}_f$ and $\mathbf{q}_p$ are the rates of filtration of the fluid Q is the crossflow of the fluid from medium p into medium f. Since the energy dissipation is governed by the filtration of the fluid in media f and p and by the crossflow between the media, the dissipative potential has the form $\Psi = \Psi(\mathbf{q}_f, \mathbf{q}_p, Q)$. In the special case, $\Psi(\mathbf{q}_f, \mathbf{q}_p, Q) = \Psi_f(\mathbf{q}_f) + \Psi_p(\mathbf{q}_p) + \Psi_Q(Q)$.

Here, Ψ_f, Ψ_p and Ψ_Q are the dissipative potentials which characterize the energy dissipation due to filtration in the media and due to the crossflow between the media, respectively. The dissipative mechanisms defined by the potentials Ψ_f, Ψ_p and Ψ_Q are assumed to be independent [4]. We shall subsequently assume that the functionals Ψ, Ψ_f, Ψ_p and Ψ_Q are convex and smooth. It follows from the principle of minimum energy dissipation in the case of steady-state processes [2, 3], that a minimum of the functional

$$I(\mathbf{q}_f, \mathbf{q}_p, Q) = \int_\Omega \Psi(\mathbf{q}_f, \mathbf{q}_p, Q)\, d\Omega \tag{1.2}$$

is attained in the real field of the variables $\mathbf{q}_f$, $\mathbf{q}_p$ and Q°. Here, the required variables must satisfy the boundary conditions and the equations of continuity (1.1).

Let us calculate the variation of the functional (1.2). This variation must be equal to zero at the point $(\mathbf{q}_f^\circ, \mathbf{q}_p^\circ, Q^\circ)$. On introducing the Lagrange multipliers λ_f and λ_p, in order to take account of the constraints (1.1), we write

$$\delta I_\lambda = \delta \left[I(\mathbf{q}_f, \mathbf{q}_p, Q) + \int_\Omega \lambda_f (\operatorname{div} \mathbf{q}_f - Q)\, d\Omega + \int_\Omega \lambda_p (\operatorname{div} \mathbf{q}_p + Q)\, d\Omega \right] \tag{1.3}$$

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After some reduction we obtain

$$\delta I_\lambda = \int_\Omega [(\partial\Psi/\partial q_{fi} - \lambda_{f,i})\,\delta q_{fi} + (\partial\Psi/\partial q_{pi} - \lambda_{p,i})\,\delta q_{pi} +$$
$$+ (\partial\Psi/\partial Q - \lambda_f + \lambda_p)\,\delta Q + (\operatorname{div}\mathbf{q}_f - Q)\,\delta\lambda_f +$$
$$+ (\operatorname{div}\mathbf{q}_p + Q)\,\delta\lambda_p]\,d\Omega + \int_\Gamma \lambda_f\,\delta q_{fn}\,d\Gamma + \int_\Gamma \lambda_p\,\delta q_{pn}\,d\Gamma \tag{1.4}$$

where q_{fn} and q_{pn} are the normal components of the velocities $\mathbf{q}_f$ and $\mathbf{q}_p$ on the boundary Γ, where Γ the boundary of the domain of the solution of the problem Ω.

It follows from the equality

$$\delta I_\lambda = 0 \tag{1.5}$$

that the coefficients λ_f and λ_p are equal to the pressures p_f and p_p, taken with the opposite sign, and that the system of equations (1.1) holds and

$$\partial\Psi/\partial q_{fi} = -p_{f,i}, \quad \partial\Psi/\partial q_{pi} = -p_{p,i}, \quad \partial\Psi/\partial Q = p_p - p_f \tag{1.6}$$

where the first two equations are the laws of filtration while the third is the equation for the crossflow. Relationships (1.6) define the structure of the generalized forces $\mathbf{X}_1 = -\nabla p_f$, $\mathbf{X}_2 = -\nabla p_p$, $\mathbf{X}_3 = p_p - p_f$. The normal rate of filtration or the equality of the pressure to zero must be specified on the boundary Γ for each medium f and p.

It is seen from (1.4) that, in order to take account of boundary conditions of the form

$$q_{fn} = q^\circ_{fn}, \; q_{pn} = q^\circ_{pn} \text{ at } \Gamma_q, \quad p_f = p^\circ_f, \; p_p = p^\circ_p \text{ at } \Gamma_q, \tag{1.7}$$

it is necessary to minimize the functional

$$I_1(\mathbf{q}_f, \mathbf{q}_p, Q) = \int_\Omega \Psi(\mathbf{q}_f, \mathbf{q}_p, Q)\,d\Omega + I_{10}, \; I_{10} = \int_{\Gamma_p} (p^\circ_f q_{fn} + p^\circ_p q_{pn})\,d\Gamma \tag{1.8}$$

instead of the functional (1.2).

We shall consider the boundary conditions in greater detail in Sec. 2. Here we merely note that the division of the boundary $\Gamma = \Gamma_q + \Gamma_p$ into the parts Γ_q and Γ_p for each medium may be made in various ways. Hence, the variational principle

$$\inf_{\mathbf{q}_f, \mathbf{q}_p, Q \in (1.1), (1.7)} I_1(\mathbf{q}_f, \mathbf{q}_p, Q) \tag{1.9}$$

is equivalent to satisfying the system of equations (1.1) and (1.6) and the boundary conditions (1.7).

On applying methods of duality [5], we obtain that the problem which is dual to the variational problem (1.9) is

$$\sup_{p_f, p_p \in (1.7)} [-I_2(p_f, p_p)] \tag{1.10}$$

that is

$$\inf_{\mathbf{q}_f, \mathbf{q}_p, Q \in (1.1), (1.7)} I_1(\mathbf{q}_f, \mathbf{q}_p, Q) = \sup_{p_f, p_p \in (1.7)} [-I_2(p_f, p_p)]$$

where

$$I_2(p_f, p_p) = \int_\Omega \Phi(\nabla p_f, \nabla p_p, p_p - p_f)\,d\Omega + I_{20}, \; I_{20} = \int_{\Gamma_q} (q^\circ_{fn} p_f + q^\circ_{pn} p_p)\,d\Gamma,$$

and $\Phi(\nabla p_f, \upsilon p_p, p_p - p_f)$ is the conjugate dissipative potential related to $\Psi(\mathbf{q}_f, \mathbf{q}_p, Q)$ by a Young–Fenhel [5] transform.

Equations (1.6) for problem (1.10) have the form

$$q_{fi} = -\partial \Phi/\partial p_{f,i}, \quad q_{pi} = -\partial \Phi/\partial p_{p,i}, \quad Q = \partial \Phi/\partial (p_p - p_f)$$

A further six variational problems, which are equivalent to the solution of the system (1.1), (1.6), (1.7) are defined by changing from problem (1.9) to the dual problem in one or two variables. Assuming, for clarity, that the dissipative mechanisms are uncoupled, we will write the following characteristic variational problems

$$\inf_{\mathbf{q}_f, Q \in (1.1),(1.7)} \sup_{p_p \in (1.7)} I_3(\mathbf{q}_f, p_p, Q), \quad \inf_{\mathbf{q}_f} \sup_{p_f, p_p \in (1.7)} I_4(\mathbf{q}_f, p_p, p_f)$$

where

$$I_3(\mathbf{q}_f, p_p, Q) = \int_\Omega [\Psi_f(\mathbf{q}_f) - \Phi_p(\nabla p_p) + \Psi_Q(Q) - Q p_p] \, d\Omega + I_{30}$$

$$I_{30} = \int_{\Gamma_p} \overset{\circ}{p}_f q_{fn} \, d\Gamma - \int_{\Gamma_q} \overset{\circ}{q}_{pn} p_p \, d\Gamma$$

$$I_4(\mathbf{q}_f, p_p, p_f) = \int_\Omega [\Psi_f(\mathbf{q}_f) - \Phi_p(\nabla p_p) - \Phi_Q(p_p - p_f) + q_f \nabla p_f] \, d\Omega - I_{20}$$

The variable Q can be eliminated in the variational problems. Problems for the functionals I_1 and I_3 using the constraint

$$\operatorname{div} \mathbf{q}_f + \operatorname{div} \mathbf{q}_p = 0 \tag{1.11}$$

can then be respectively written in the form

$$\inf_{\mathbf{q}_f, \mathbf{q}_p \in (1.7),(1.11)} I_1'(\mathbf{q}_f, \mathbf{q}_p), \quad \inf_{\mathbf{q}_f \in (1.7)} \sup_{p_p \in (1.7)} I_3'(\mathbf{q}_f, p_p)$$

where

$$I_1'(\mathbf{q}_f, \mathbf{q}_p) = \int_\Omega [\Psi_f(\mathbf{q}_f) + \Psi_p(\mathbf{q}_p) + \Psi_Q(\operatorname{div} \mathbf{q}_f)] \, d\Omega + I_{10}$$

$$I_3'(\mathbf{q}_f, p_p) = \int_\Omega [\Psi_f(\mathbf{q}_f) - \Phi_p(\nabla p_p) + \Psi(\operatorname{div} \mathbf{q}_f) - p_p \operatorname{div} \mathbf{q}_f] \, d\Omega + I_{30}$$

2. We will now construct the variational principles of two-phase filtration and write the equations of continuity in medium f, in medium p, the relation between the pressures in the phases and the expression for the derivative of the entropy

$$-\operatorname{div} \mathbf{q}_{f1} = m_f s_{f,t} - Q_1, \quad -\operatorname{div} \mathbf{q}_{f2} = -m_f s_{f,t} - Q_2 \tag{2.1}$$

$$-\operatorname{div} \mathbf{q}_{p1} = m_p s_{p,t} + Q_1, \quad -\operatorname{div} \mathbf{q}_{p2} = -m_p s_{p,t} + Q_2 \tag{2.2}$$

$$p_{f1} - p_{f2} = p_{fc}(s_f), \quad p_{p1} - p_{p2} = p_{pc}(s_p) \tag{2.3}$$

$$\sigma = \mathbf{XY} = \mathbf{X}_1 \mathbf{Y}_1 + \ldots + \mathbf{X}_6 \mathbf{Y}_6$$

Here m_f and m_p are the porosities, Q_1 and Q_2 are the crossflows of the first and second phases from medium p into medium f, s_f and s_p are the saturations of the first phase, $\mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_6)$ are generalized forces, $\mathbf{Y} = (\mathbf{Y}_1, \ldots, \mathbf{Y}_6)$ are generalized velocities; $\mathbf{Y}_1 = q_{f1}$,

$\mathbf{Y}_2 = q_{f2}$, $\mathbf{Y}_3 = q_{p1}$, $\mathbf{Y}_4 - q_{p2}$, $\mathbf{Y}_5 = Q_1$, $\mathbf{Y}_6 = Q_2$; $\mathbf{X}_1 = -\nabla p_{f1}$, $\mathbf{X}_2 = -\nabla p_{f2}$, $\mathbf{X}_3 = -\nabla p_{p1}$, $\mathbf{X}_4 = -\nabla p_{p2}$, $\mathbf{X}_5 = p_{p1} - p_{f1}$, $\mathbf{X}_6 = p_{p2} - p_{f2}$. Within the framework of the hypothesis of normal dissipation [4, 6], a dissipation potential $\Psi(\mathbf{Y})$, exists, that is, a convex characteristic functional which is semicontinuous from below such that

$$\mathbf{X} \in \partial \Psi(\mathbf{Y}) \tag{2.4}$$

where $\mathbf{X}$ is the subgradient of the functional $\Psi(\mathbf{Y})$ at the point $\mathbf{Y}$. The inverse relationship [6, 7]

$$\mathbf{Y} \in \partial \Phi(\mathbf{X}) \tag{2.5}$$

follows from (2.4), where $\Phi(\mathbf{X})$ is the conjugate dissipation potential. Any of the relationships (2.4), (2.5) closes the system of equations (2.1)–(2.3).

The following assertions are equivalent [6, 7]: (1) $\mathbf{X}' \in \partial\Psi(\mathbf{Y}')$, (2) $\Psi(\mathbf{Y}) - \mathbf{X}'\mathbf{Y}$ attains a minimum with respect to $\mathbf{Y}$ at the point $\mathbf{Y} = \mathbf{Y}'$, (3) $\mathbf{Y}' \in \partial\Phi(\mathbf{X}')$, and (4) $\partial(\mathbf{X}) - \mathbf{X}\mathbf{Y}'$ attains a minimum with respect to $\mathbf{X}$ at the point $\mathbf{X} = \mathbf{X}'$. They are the basis for constructing the variational principles.

We shall henceforth assume the functionals $\Psi(\mathbf{Y})$, $\Phi(\mathbf{X})$ to be smooth

$$\mathbf{X} = \operatorname{grad} \Psi(\mathbf{Y}), \quad \mathbf{Y} = \operatorname{grad} \Phi(\mathbf{X}) \tag{2.6}$$

Let us write the system of equations (2.1)–(2.3), (2.6) in the form

$$\mathbf{q}_{fk} = -\partial \Phi / \partial \nabla p_{fk} \quad \text{or} \quad -\nabla p_{fk} = \partial \Psi / \partial \mathbf{q}_{fk}$$

$$\mathbf{q}_{pk} = -\partial \Phi / \partial \nabla p_{pk} \quad \text{or} \quad -\nabla p_{pk} = \partial \Psi / \partial \mathbf{q}_{pk}$$

$$Q_k = \partial \Phi / \partial (p_{pk} - p_{fk}) \quad \text{or} \quad p_{pk} - p_{fk} = \partial \Psi / \partial Q_k, \quad k = 1, 2 \tag{2.7}$$

$$p_{f1} - p_{f2} = p_{fc}(s_f), \quad p_{p1} - p_{p2} = p_{pc}(s_p), \quad \operatorname{div} \mathbf{q}_f = Q, \quad \operatorname{div} \mathbf{q}_p = -Q \tag{2.8}$$

$$-\operatorname{div} \mathbf{q}_{f1} = m_f s_{f,t} - Q_1 \quad \text{or} \quad -\operatorname{div} \mathbf{q}_{f2} = -m_f s_{f,t} - Q_2 \tag{2.9}$$

$$-\operatorname{div} \mathbf{q}_{p1} = m_p s_{p,t} + Q_1 \quad \text{or} \quad -\operatorname{div} \mathbf{q}_{p2} = -m_p s_{p,t} + Q_2$$

$$(Q = Q_1 + Q_2, \; \mathbf{q}_f = \mathbf{q}_{f1} + \mathbf{q}_{f2}, \; \mathbf{q}_p = \mathbf{q}_{p1} + \mathbf{q}_{p2})$$

We will now construct a variational principle in the generalized velocities. It is seen from assertions (1)–(4) that, in the case of a process $(\mathbf{X}^\circ, \mathbf{Y}^\circ)$ which actually occurs in the domain Ω, the value of $\mathbf{Y}^\circ$, which corresponds to $\mathbf{X}^\circ$, is determined from the solution of the problem

$$\inf_{\mathbf{Y}} B_1(\mathbf{Y}) = \inf_{\mathbf{Y}} \int_\Omega [\Psi(\mathbf{Y}) - \mathbf{X}^\circ \mathbf{Y}] \, d\Omega \tag{2.10}$$

which is equivalent to satisfying the governing relationships (2.7) and corresponds to the principle of minimum energy dissipation [2, 3] in general form. We will use variational problem (2.10) to construct a variational principle in which it should be sufficient to use boundary values of the quantity $\mathbf{X}^\circ$ instead of a knowledge of $\mathbf{X}^\circ$ in the whole domain Ω to find $\mathbf{Y}^\circ$. Let us transform the integral $\int_\Omega \mathbf{X}^\circ \mathbf{Y} \, d\Omega$, using the expressions for the pressures

$$p_f = l_f p_{1f} + (1 - l_f) p_{2f}, \quad p_p = l_p p_{1p} + (1 - l_p) p_{2p}$$

where l_+ can be equated to zero, unity, the saturation s_f or the Backley–Leverett function, and likewise in the case of l_p. As a result, we obtain the functional

$$I_1(\mathbf{Y}) = \int_\Omega [\Psi(\mathbf{Y}) + \nabla((1 - l_f) p_{fc}) \mathbf{q}_{f1} + \nabla((1 - l_p) p_{pc}) \mathbf{q}_{p1} -$$

$$- \nabla(l_f p_{fc}) \mathbf{q}_{f2} - \nabla(l_p p_{pc}) \mathbf{q}_{p2} - Q_1 ((1 - l_p) p_{pc} - (1 - l_f) p_{fc}) +$$

$$+ Q_2 (l_p p_{pc} - l_f p_{fc})] \, d\Omega + I_{10} \tag{2.11}$$

Evaluating the minimum of functional (2.11), subject to constraints corresponding to the last two relationships in (2.8) and

$$q_{fn} = q^\circ_{fn}, \quad q_{pn} = q^\circ_{pn} \text{ on } \Gamma_q \tag{2.12}$$

is equivalent to solving the system of equations (2.7), (2.8) with boundary conditions (2.12) and

$$p_f = p^\circ_f, \quad p_p = p^\circ_p \text{ on } \Gamma_p \tag{2.13}$$

Similarly, from the problem

$$\inf_{\mathbf{X}} B_2 (\mathbf{X}) = \inf_{\mathbf{X}} \int_\Omega [\Phi (\mathbf{X}) - \mathbf{X}\mathbf{Y}^\circ] \, d\Omega \tag{2.14}$$

we construct the variational principle in the pressures

$$\inf_{p_f, p_p \in (2.13)} I_2 (p_f, p_p) = \inf_{p_f, p_p \in (2.13)} [\int \Phi (\mathbf{X}) \, d\Omega + I_{20}] \tag{2.15}$$

The equality

$$\inf_{\mathbf{Y} \in (2.8), (2.12)} I_1 (\mathbf{Y}) = \sup_{p_f, p_p \in (2.13)} [-I_2 (p_f, p_p)]$$

holds.

Problem (2.15) is dual to the problem

$$\inf_{\mathbf{Y} \in (2.8), (2.12)} I_1 (\mathbf{Y}) \tag{2.16}$$

with respect to all of the variables $\mathbf{Y}_1, \ldots, \mathbf{Y}_6$. On passing from problem (2.16) to the dual problem with respect to the different groups of variables $\mathbf{Y}_1, \ldots, \mathbf{Y}_6$. we obtain a whole set of dual variational problems.

In the case of linear laws of filtration (d'Arcy's laws) and linear laws for the crossflows

$$\mathbf{q}_{fk} = -(k_f/\mu_k) f_{fk} (s_f) \nabla p_{fk}, \quad \mathbf{q}_{pk} = -(k_p/\mu_k) f_{pk} (s_p) \nabla p_{pk}$$
$$Q_k = b_k (p_{pk} - p_{fk}) + c_k, \quad k = 1, 2$$

which corresponds to quadratic functionals Ψ_f, Ψ_p and Ψ_Q, problem (2.16) reduces to the form

$$\inf_{\mathbf{q}_f, \mathbf{q}_p, Q \in (2.8), (2.12)} I_1 (\mathbf{q}_f, \mathbf{q}_p Q) \tag{2.17}$$

where

$$I_1 (\mathbf{q}_f, \mathbf{q}_p, Q) = \int_\Omega [\Psi_f (\mathbf{q}_f) + \Psi_p (\mathbf{q}_p) + \Psi_Q (Q)] \, d\Omega + I_{10}$$
$$\Psi_f (\mathbf{q}_f) = \frac{1}{2} \frac{\mu_1}{k_f \varphi_f (s_f)} | \mathbf{q}_f |^2 + [F_f (s_f) \nabla p_{fc} - \nabla (l_f p_{fc})] \, \mathbf{q}_f$$
$$\Psi_Q (Q) = (b_1 + b_2)^{-1} (Q^2/2 - [b_1 ((1-l_p) p_{pc} - (1-l_f) p_{fc}) +$$
$$+ b_2 (l_f p_{fc} - l_p p_{pc}) + (c_1 + c_2)] \, Q)$$
$$\varphi_f (s_f) = f_{f1} (s_f) + (\mu_1/\mu_2) f_{f2} (s_f), \quad F_f (s_f) = f_{f1} (s_f)/\varphi_f (s_f)$$
$$b_k = b_k (s_f, s_p), \quad c_k = c_k (s_f, s_p)$$

k_f and k_p are the absolute permeabilities, μ_1 and μ_2 are the viscosities and $f_{fk}(s_f)$, $f_{pk}(s_p)$ ($k=1$, 2) are the relative phase permeabilities. The expressions for $\Psi(\mathbf{q}_p)$, $\varphi_p(s_p)$ and $F_p(s_p)$ have an analogous form.

The variational principle (2.17) is equivalent to solving a system of equations which has the form of (1.1), (1.6) with boundary conditions (1.7). Hence, all the dual variational principles have a form which is similar to the variational principles of steady single-phase filtration.

The variational principles which have been obtained enable us to determine the velocity fields of the filtration and the crossflows, and the pressures at fixed saturations. Equations (2.9) serve to take account of the change in s_f and s_p.

No knowledge of the boundary conditions, which are necessary for solving the problem, is required when deriving the variational principles. The combinations of the required variables, specified on the boundary, for which a solution of the problem exists, are established from an analysis of the boundary integrals in the variational principles which are obtained by transforming the initial variational problems (2.1) and (2.14).

As an example the integral I_{20} in variational principle (2.15), which is obtained from problem (2.14) ignoring the boundary conditions, must be taken over the entire boundary Γ that corresponds to the boundary conditions $\mathbf{q}_{fn}|_\Gamma=\mathbf{q}^\circ_{fn}$ and $\mathbf{q}_{pn}|_\Gamma=\mathbf{q}^\circ_{pn}$ being satisfied. When account is taken of the boundary conditions (2.12) and (2.13), the integral I_{20} must be taken over the part of the boundary Γ_q, and the variables p_f and p_p satisfy condition (2.13), which is seen from the representation

$$\int_\Gamma (q^\circ_{fn}p_f+q^\circ_{pn}p_p)\,d\Gamma = \int_{\Gamma_q}(q^\circ_{fn}p_f+q^\circ_{pn}p_p)\,d\Gamma+\int_{\Gamma_p}(q^\circ_{fn}p^\circ_f+q^\circ_{pn}p^\circ_p)\,d\Gamma \tag{2.18}$$

The integral with respect to Γ_p in (2.18) is omitted as a constant quantity. When the equality $p_f=p_p=p$ on Γ_p holds, it is sufficient, when obtaining the solution, to specify the normal component of the velocity $\mathbf{q}=\mathbf{q}_f+\mathbf{q}_p$

$$I_{20}=\int_{\Gamma_q} q^\circ_n p\,d\Gamma$$

When $p_f=p_p=p=\text{const}$ on Γ_q, the specification of the flow rate G° on Γ_q is sufficient for obtaining the solution

$$I_{20}=pG^\circ$$

where p is a quantity which is unknown though constant on Γ_q.

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